

# Modified equipartition calculation for supernova remnants

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## ABSTRACT

Determination of the magnetic field strength in the interstellar medium is one of the most complex tasks of contemporary astrophysics. We can only estimate the order of magnitude of the magnetic field strength by using a few very limited methods. Besides Zeeman effect and Faraday rotation, the equipartition or the minimum-energy calculation is a widespread method for estimating magnetic field strength and energy contained in the magnetic field and cosmic ray particles by using only the radio synchrotron emission. Despite of its approximate character, it remains a useful tool, especially when there is no other data about the magnetic field in a source. In this paper we give a modified calculation which we think is more appropriate for estimating magnetic field strengths and energetics in supernova remnants (SNRs). Finally, we present calculated estimates of the magnetic field strengths for all Galactic SNRs for which the necessary observational data are available. The web application for calculation of the magnetic field strength of SNRs is available at <http://poincare.matf.bg.ac.rs/~arbo/eqp/>.

*Subject headings:* ISM: magnetic fields — supernova remnants — radio continuum: general

## 1. Introduction

The basic constituents of the interstellar medium (ISM) are: normal (thermalized) particles, cosmic rays (CRs), radiation and magnetic field. Each of these four forms of ISM contains similar energy density of about 1 eV/cm<sup>3</sup>. If we compare quantity of information

available for each of them, we can immediately conclude that the magnetic field is absolutely the most intrigued and hidden form of ISM. Recent simulations of SNR shocks commonly include magnetic field because it plays an important part in various related phenomena (particle acceleration, radiation, shock compression and formation, etc). The magnetic field strength and its direction can only be approximately estimated by using a few, in their applicabilities, very limited methods (for recent review of magnetic fields in supernova remnants see Reynolds et al. 2011). One of them is Zeeman effect - it is appropriate method for generally stronger fields - it can be used for determination of strong ISM magnetic fields in high density HI or molecular clouds rich with OH and CN. The global magnetic field of the Galaxy, a few  $\mu\text{G}$ , is too small to be measured in this way. The second method for determination of the component of ISM magnetic field parallel to the line of sight is so-called Faraday rotation or rotation measure method. Rotation measure (RM) is calculated directly from the radio astronomical polarization observations at multiple frequencies. This quantity depends on the plasma density and the strength of the field component along the line of sight. Under necessary simplistic assumptions RM can yield an order of magnitude estimate of the magnetic field strength between the source and observer. If several distinct rotating regions located along the line of sight generate a spectrum of various RM components, multi-channel spectro-polarimetric radio data are needed that can be Fourier-transformed into Faraday space, called RM synthesis (see Heald 2009, Beck 2011 and references therein). If we would like to estimate the magnetic field strength directly connected to a source embedded in the relatively low density region, the only way is by using the so-called equipartition calculation.

The equipartition or the minimum-energy calculation is a widespread method for estimating magnetic field strength and energy contained in the magnetic field and cosmic ray particles by using only the radio synchrotron emission of a source. Despite of its approximate character, it remains a useful tool in situations when no other data about the source are available. Details of equipartition and revised equipartition calculations for radio sources in general are available in Pacholczyk (1970, hereafter P70), Govoni & Feretti (2004), and Beck & Krause (2005, hereafter BK05), respectively. A discussion on whether equipartition of energy is fulfilled in real sources, and how reliable magnetic field estimates from equipartition calculation are, can be found in Duric (1990).

In his famous book, Pacholczyk gave fundamental concepts of the equipartition or the minimum-energy calculation. The first ingredient of the equipartition calculation is expression for total energy of relativistic particles, which can be obtained by integration of power-law energy distribution of cosmic rays. Total energy of relativistic particles was found by integration over all frequencies in the radio domain. Pacholczyk assumed homogenous magnetic field for calculation of energy contained in the magnetic field, and coefficient  $k$

which represents ratio between energies of relativistic protons and electrons. Finally, the last ingredient in the P70 equipartition formula is the radio luminosity of an object.

BK05 presented the revised equipartition calculation. The basic improvement in comparison to the classical, P70 equipartition, is integration of power-law energy distribution over energies instead over frequencies. They integrated over two energy ranges with a break at  $E = mc^2$  where  $m$  is the rest mass of the accelerated particles, i.e. two power-law distributions with different slopes, both dependent on energy spectral index  $\gamma$ . Instead of luminosity used in the classical approach, BK05 used radio intensity - their intention was to determine the magnetic field strength of the small part of the very extended objects such as whole Galaxy or an extragalactic system. The magnetic field small scale structures of very extended objects are very far from being homogenous. The model of magnetic field distribution used in the revised equipartition formula is accommodated for the previously described objects. Finally, BK05 used coefficient  $\mathbf{K}_0$  which represents ratio of the number densities of cosmic ray protons to electrons, instead of ratio between energies of protons and electrons used in the classical equipartition.

In this paper, we use the energy ratio, as in the classical calculation, but it includes all heavier particles which can be found in cosmic rays. Also, we use the radio flux density instead the radio luminosity as in P70 equipartition, or the specific intensity from revised calculation of BK05. Since our intention is to derive equipartition formulae for the determination of the magnetic fields and the minimal energies in supernova remnants (SNRs), we use model of the magnetic field distribution defined in Longair (1994). Finally, since the distribution of CRs is a power-law in momentum (which can be transformed to the same power-law in energy, for energies high enough), we have chosen to integrate over momentum and not over energies as BK05 did, so there is no need for introduction of the break in the differential energy spectrum.

We emphasize that the final formulae in the P70 equipartition do not depend on the energy spectral index (or radio spectral index,  $\alpha = (\gamma - 1)/2$ ), while in the BK05 and our equipartition these formulae depend on the energy spectral index (see equations (12) and (13)).

In the next section, by relying on Bell's theory of diffusive shock acceleration - DSA, (Bell 1978a,b), and his assumption concerning injection of particles into the acceleration process, we will first derive a modified equipartition i.e. minimum-energy calculation (Arbutina et al. 2011) applicable to 'mature' SNRs ( $v_s \ll 6000 - 7000$  km/s) with radio spectral index  $0.5 < \alpha < 1$  (energy spectral index  $2 < \gamma < 3$ ). Then we will incorporate the dependence  $\epsilon = \epsilon(E_{\text{inj}})$  which will make formula applicable to the younger i.e. all SNRs.

## 2. Analysis and Results

### 2.1. A simple approach

Following Bell (1978b) we will assume that a certain number of particles have been injected into the acceleration process all with the same injection energy  $E_{inj} \approx 4\frac{1}{2}m_p v_s^2$ .<sup>1</sup> If we assume that shock velocity is low enough so that  $E_{inj} \ll m_e c^2$  (and  $p_{inj}^e \ll m_e c$ ), for energy density of a cosmic ray species (e.g. electrons, protons,  $\alpha$ -particles, heavier ions), assuming power-law momentum distribution, we have

$$\begin{aligned}
 \epsilon &= \int_{p_{inj}}^{p_\infty} 4\pi k p^{-\gamma} (\sqrt{p^2 c^2 + m^2 c^4} - m c^2) dp \\
 &\approx \int_0^\infty 4\pi k p^{-\gamma} (\sqrt{p^2 c^2 + m^2 c^4} - m c^2) dp \\
 &= 4\pi k c (m c)^{2-\gamma} \int_0^\infty x^{-\gamma} (\sqrt{x^2 + 1} - 1) dx, \quad x = \frac{p}{m c} \\
 &= K (m c^2)^{2-\gamma} \frac{\Gamma(\frac{3-\gamma}{2}) \Gamma(\frac{\gamma-2}{2})}{2\sqrt{\pi}(\gamma-1)}, \quad K = 4\pi k c^{\gamma-1}, \quad 2 < \gamma < 3.
 \end{aligned} \tag{1}$$

where  $k$  is the constant in the distribution function  $f(p) = k p^{-(\gamma+2)}$ . Function under the integral in equation (1) is approximately a power-law with a spectral index of  $2 - \gamma$  for thermal (non-relativistic) particles and a power-law with a spectral index of  $1 - \gamma$  for highly relativistic particles. In this paper the sharp break in BK05 is replaced by a smooth one.

Total cosmic ray energy density is then

$$\begin{aligned}
 \epsilon_{CR} &= \frac{\Gamma(\frac{3-\gamma}{2}) \Gamma(\frac{\gamma-2}{2})}{2\sqrt{\pi}(\gamma-1)} \left( K_e (m_e c^2)^{2-\gamma} + \sum_i K_i (m_i c^2)^{2-\gamma} \right) \\
 &= \frac{\Gamma(\frac{3-\gamma}{2}) \Gamma(\frac{\gamma-2}{2})}{2\sqrt{\pi}(\gamma-1)} \left( K_e (m_e c^2)^{2-\gamma} + K_p (m_p c^2)^{2-\gamma} \sum_i \frac{n_i}{n_p} \left( \frac{m_i}{m_p} \right)^{(3-\gamma)/2} \right) \\
 &= \frac{\Gamma(\frac{3-\gamma}{2}) \Gamma(\frac{\gamma-2}{2})}{2\sqrt{\pi}(\gamma-1)} K_e (m_e c^2)^{2-\gamma} \left( 1 + \frac{n}{n_e} \left( \frac{m_p}{m_e} \right)^{(3-\gamma)/2} \sum_i \frac{n_i}{n} \left( \frac{m_i}{m_p} \right)^{(3-\gamma)/2} \right) \\
 &= K_e (m_e c^2)^{2-\gamma} \frac{\Gamma(\frac{3-\gamma}{2}) \Gamma(\frac{\gamma-2}{2})}{2\sqrt{\pi}(\gamma-1)} (1 + \kappa),
 \end{aligned} \tag{2}$$

where

$$\kappa = \left( \frac{m_p}{m_e} \right)^{(3-\gamma)/2} \frac{\sum_i A_i^{(3-\gamma)/2} \nu_i}{\sum_i Z_i \nu_i}, \tag{3}$$

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<sup>1</sup>We assume fully ionized, globally electro-neutral plasma.

$\kappa$  represents the energy ratio between ions and electrons,  $n_e = \sum_i Z_i n_i$ ,  $\nu_i = n_i/n$  are ion abundances,  $A_i$  and  $Z_i$  are mass and charge numbers of elements and we assumed that at high energies  $K_p/K_e \approx (n_p/n_e) \cdot (m_p/m_e)^{(\gamma-1)/2}$  (see equation (26)), where  $K_p$  and  $K_e$  are the constants in the power-law energy distributions for protons and electrons, respectively. Note that we have neglected energy losses.

Emission coefficient for synchrotron radiation is, on the other hand,

$$\varepsilon_\nu = c_5 K_e (B \sin \Theta)^{(\gamma+1)/2} \left( \frac{\nu}{2c_1} \right)^{(1-\gamma)/2}, \quad (4)$$

where  $c_1, c_3$  and  $c_5 = c_3 \Gamma(\frac{3\gamma-1}{12}) \Gamma(\frac{3\gamma+19}{12}) / (\gamma+1)$  are defined in P70.<sup>2</sup> We will use the flux density which is defined as

$$S_\nu = \frac{L_\nu}{4\pi d^2} = \frac{\mathcal{E}_\nu V}{4\pi d^2} = \frac{\frac{4\pi}{3} R^3 f \mathcal{E}_\nu}{4\pi d^2} = \frac{4\pi}{3} \varepsilon_\nu f \theta^3 d, \quad (5)$$

where  $L_\nu$  is radio luminosity,  $\mathcal{E}_\nu$  is volume emisivity,  $V$  is the volume,  $f$  is volume filling factor of radio emission,  $R$  is the radius,  $d$  is the distance and  $\theta = R/d$  is angular radius.

If we assume isotropic distribution for the orientation of pitch angles (Longair 1994) we can take for the average  $\langle (\sin \Theta)^{(\gamma+1)/2} \rangle$

$$\frac{1}{2} \int_0^\pi (\sin \Theta)^{(\gamma+3)/2} d\Theta = \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{\gamma+5}{4})}{\Gamma(\frac{\gamma+7}{4})}. \quad (6)$$

For the total energy we have

$$E = \frac{4\pi}{3} R^3 f (\epsilon_{\text{CR}} + \epsilon_B), \quad \epsilon_B = \frac{1}{8\pi} B^2, \quad (7)$$

$$E = \frac{4\pi}{3} R^3 f \left( K_e (m_e c^2)^{2-\gamma} \frac{\Gamma(\frac{3-\gamma}{2}) \Gamma(\frac{\gamma-2}{2})}{2\sqrt{\pi}(\gamma-1)} (1+\kappa) + \frac{1}{8\pi} B^2 \right). \quad (8)$$

Looking for the minimum energy with respect to  $B$ ,  $\frac{dE}{dB} = 0$  gives

$$\frac{dK_e}{dB} (m_e c^2)^{2-\gamma} \frac{\Gamma(\frac{3-\gamma}{2}) \Gamma(\frac{\gamma-2}{2})}{2\sqrt{\pi}(\gamma-1)} (1+\kappa) + \frac{1}{4\pi} B = 0, \quad (9)$$

where (by using (4), (5) and (6))

$$\frac{dK_e}{dB} = -\frac{3}{4\pi} \frac{S_\nu}{f \theta^3 d} \frac{1}{c_5} \left( \frac{\nu}{2c_1} \right)^{-(1-\gamma)/2} (\gamma+1) \frac{\Gamma(\frac{\gamma+7}{4})}{\sqrt{\pi} \Gamma(\frac{\gamma+5}{4})} B^{-(\gamma+3)/2}, \quad (10)$$

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<sup>2</sup>Namely,  $c_1 = 6.264 \cdot 10^{18}$  and  $c_3 = 1.866 \cdot 10^{-23}$  in cgs units.

i.e. the magnetic field for the minimum energy is

$$B = \left( \frac{3}{2\pi} \frac{(\gamma+1)\Gamma(\frac{3-\gamma}{2})\Gamma(\frac{\gamma-2}{2})\Gamma(\frac{\gamma+7}{4})}{(\gamma-1)\Gamma(\frac{\gamma+5}{4})} \frac{S_\nu}{f d \theta^3} \cdot (m_e c^2)^{2-\gamma} \frac{(2c_1)^{(1-\gamma)/2}}{c_5} (1+\kappa) \nu^{(\gamma-1)/2} \right)^{2/(\gamma+5)}, \quad (11)$$

or

$$B \text{ [G]} \approx \left( 6.286 \cdot 10^{(9\gamma-79)/2} \frac{\gamma+1}{\gamma-1} \frac{\Gamma(\frac{3-\gamma}{2})\Gamma(\frac{\gamma-2}{2})\Gamma(\frac{\gamma+7}{4})}{\Gamma(\frac{\gamma+5}{4})} (m_e c^2)^{2-\gamma} \cdot \frac{(2c_1)^{(1-\gamma)/2}}{c_5} (1+\kappa) \frac{S_\nu [\text{Jy}]}{f d [\text{kpc}] \theta [\text{arcmin}]^3} \nu [\text{GHz}]^{(\gamma-1)/2} \right)^{2/(\gamma+5)}, \quad (12)$$

where  $m_e c^2 \approx 8.187 \cdot 10^{-7}$  ergs. We also have

$$E_B = \frac{\gamma+1}{4} E_{\text{CR}}, \quad E_{\text{min}} = \frac{\gamma+5}{\gamma+1} E_B. \quad (13)$$

This result is the same as in BK05.

## 2.2. A more general formula for $\kappa$

Let us start again with equation (1)

$$\begin{aligned} \epsilon &\approx \int_{p_{\text{inj}}}^{\infty} 4\pi k p^{-\gamma} (\sqrt{p^2 c^2 + m^2 c^4} - m c^2) dp \\ &= 4\pi k c (m c)^{2-\gamma} \int_{\frac{p_{\text{inj}}}{m c}}^{\infty} x^{-\gamma} (\sqrt{x^2 + 1} - 1) dx, \quad x = \frac{p}{m c} \\ &= 4\pi k c (m c)^{2-\gamma} I\left(\frac{p_{\text{inj}}}{m c}\right). \end{aligned} \quad (14)$$

Integral  $I(x)$  can be expressed through Gauss hypergeometric function  ${}_2F_1$  (for  $\gamma > 2$ )

$$I(x) = \frac{\Gamma(\frac{3-\gamma}{2})\Gamma(\frac{\gamma-2}{2})}{2\sqrt{\pi}(\gamma-1)} - \frac{x^{1-\gamma}(1 - {}_2F_1(-\frac{1}{2}, \frac{1-\gamma}{2}, \frac{3-\gamma}{2}; -x^2))}{\gamma-1}, \quad (15)$$

but we will try to find more simple approximation. First notice that

$$I(x) \approx \frac{\Gamma(\frac{3-\gamma}{2})\Gamma(\frac{\gamma-2}{2})}{2\sqrt{\pi}(\gamma-1)} - \frac{x^{3-\gamma}}{2(3-\gamma)} + \frac{x^{5-\gamma}}{8(5-\gamma)} - \dots, \quad x \rightarrow 0, \quad (16)$$

$$I(x) \approx \frac{x^{2-\gamma}}{\gamma-2}, \quad x \rightarrow \infty. \quad (17)$$

So we can try an approximation ( $2 < \gamma < 3$ )

$$I(x)_{\text{approx}} = \frac{\frac{\Gamma(\frac{3-\gamma}{2})\Gamma(\frac{\gamma-2}{2})}{2\sqrt{\pi}(\gamma-1)} - \frac{x^{3-\gamma}}{2(3-\gamma)} + F(\gamma)x^{5-\gamma}}{1 + F(\gamma)(\gamma-2)x^3} \quad (18)$$

which has correct limits when  $x \rightarrow 0$  and  $x \rightarrow \infty$ . We shall find  $F(\gamma)$  from matching condition  $I(1) = I(1)_{\text{approx}}$ :

$$F(\gamma) = \frac{\frac{1}{2(3-\gamma)} - \frac{1 - {}_2F_1(-\frac{1}{2}, \frac{1-\gamma}{2}, \frac{3-\gamma}{2}; -1)}{\gamma-1}}{1 - (\gamma-2)\left(\frac{\Gamma(\frac{3-\gamma}{2})\Gamma(\frac{\gamma-2}{2})}{2\sqrt{\pi}(\gamma-1)} - \frac{1 - {}_2F_1(-\frac{1}{2}, \frac{1-\gamma}{2}, \frac{3-\gamma}{2}; -1)}{\gamma-1}\right)}. \quad (19)$$

Since the last expression also involves hypergeometric function we found by trial and error an approximation

$$F(\gamma)_{\text{approx}} = \frac{17}{1250} \frac{(2\gamma+1)\gamma}{(\gamma-2)(5-\gamma)} \quad (20)$$

From now on we will assume  $I(x) = I(x)_{\text{approx}}$  and  $F(\gamma) = F(\gamma)_{\text{approx}}$  (relative error is less than 3.5 %).

Total cosmic rays energy density is then

$$\epsilon_{\text{CR}} = \epsilon_e + \epsilon_{\text{ion}} = K_e(m_e c^2)^{2-\gamma} I\left(\frac{p_{\text{inj}}^e}{m_e c}\right) + \sum_i K_i(m_i c^2)^{2-\gamma} I\left(\frac{p_{\text{inj}}^i}{m_i c}\right), \quad (21)$$

where (because  $\frac{p_{\text{inj}}^i}{m_i c} \ll 1$ )

$$\begin{aligned} \epsilon_{\text{ion}} &\approx \sum_i K_i(m_i c^2)^{2-\gamma} \left( \frac{\Gamma(\frac{3-\gamma}{2})\Gamma(\frac{\gamma-2}{2})}{2\sqrt{\pi}(\gamma-1)} - \frac{1}{2(3-\gamma)} \left( \frac{\sqrt{E_{\text{inj}}^2 + 2m_i c^2 E_{\text{inj}}}}{m_i c^2} \right)^{3-\gamma} \right) \\ &\approx K_p(m_p c^2)^{2-\gamma} \sum_i \frac{n_i}{n_p} \left( \frac{p_{\text{inj}}^i}{p_{\text{inj}}^p} \right)^{\gamma-1} \left( \frac{m_i}{m_p} \right)^{2-\gamma} \cdot \\ &\quad \cdot \left( \frac{\Gamma(\frac{3-\gamma}{2})\Gamma(\frac{\gamma-2}{2})}{2\sqrt{\pi}(\gamma-1)} - \frac{1}{2(3-\gamma)} \left( \frac{2E_{\text{inj}}}{m_i c^2} \right)^{(3-\gamma)/2} \right) \\ &\approx K_p(m_p c^2)^{2-\gamma} \sum_i \left[ \frac{n_i}{n_p} \left( \frac{m_i}{m_p} \right)^{(3-\gamma)/2} \frac{\Gamma(\frac{3-\gamma}{2})\Gamma(\frac{\gamma-2}{2})}{2\sqrt{\pi}(\gamma-1)} - \frac{1}{2(3-\gamma)} \left( \frac{2E_{\text{inj}}}{m_p c^2} \right)^{(3-\gamma)/2} \frac{n_i}{n_p} \right] \\ &\approx K_p(m_p c^2)^{2-\gamma} \frac{n}{n_p} \left[ \frac{\Gamma(\frac{3-\gamma}{2})\Gamma(\frac{\gamma-2}{2})}{2\sqrt{\pi}(\gamma-1)} \sum_i A_i^{(3-\gamma)/2} \nu_i - \frac{1}{2(3-\gamma)} \left( \frac{2E_{\text{inj}}}{m_p c^2} \right)^{(3-\gamma)/2} \right]. \quad (22) \end{aligned}$$

Finally

$$\epsilon_{\text{CR}} = K_e(m_e c^2)^{2-\gamma} \left[ I\left( \frac{\sqrt{E_{\text{inj}}^2 + 2m_e c^2 E_{\text{inj}}}}{m_e c^2} \right) + \frac{1}{\sum_i Z_i \nu_i} \left( \frac{m_p}{m_e} \right)^{2-\gamma} \left( \frac{2m_p c^2 E_{\text{inj}}}{E_{\text{inj}}^2 + 2m_e c^2 E_{\text{inj}}} \right)^{(\gamma-1)/2} \right].$$

$$\begin{aligned}
& \cdot \left( \frac{\Gamma(\frac{3-\gamma}{2})\Gamma(\frac{\gamma-2}{2})}{2\sqrt{\pi}(\gamma-1)} \sum_i A_i^{(3-\gamma)/2} \nu_i - \frac{1}{2(3-\gamma)} \left( \frac{2E_{\text{inj}}}{m_p c^2} \right)^{(3-\gamma)/2} \right) \Big] \\
& = K_e (m_e c^2)^{2-\gamma} \frac{\Gamma(\frac{3-\gamma}{2})\Gamma(\frac{\gamma-2}{2})}{2\sqrt{\pi}(\gamma-1)} (1 + \kappa), \tag{23}
\end{aligned}$$

where

$$\begin{aligned}
\kappa & = I \left( \frac{\sqrt{E_{\text{inj}}^2 + 2m_e c^2 E_{\text{inj}}}}{m_e c^2} \right) \left( \frac{\Gamma(\frac{3-\gamma}{2})\Gamma(\frac{\gamma-2}{2})}{2\sqrt{\pi}(\gamma-1)} \right)^{-1} + \frac{1}{\sum_i Z_i \nu_i} \left( \frac{m_p}{m_e} \right)^{2-\gamma} \left( \frac{2m_p c^2 E_{\text{inj}}}{E_{\text{inj}}^2 + 2m_e c^2 E_{\text{inj}}} \right)^{(\gamma-1)/2} \cdot \\
& \cdot \left( \sum_i A_i^{(3-\gamma)/2} \nu_i - \frac{1}{2(3-\gamma)} \left( \frac{2E_{\text{inj}}}{m_p c^2} \right)^{(3-\gamma)/2} \left( \frac{\Gamma(\frac{3-\gamma}{2})\Gamma(\frac{\gamma-2}{2})}{2\sqrt{\pi}(\gamma-1)} \right)^{-1} \right) - 1. \tag{24}
\end{aligned}$$

In the above derivation we used the fact that (Bell 1978b)

$$K_i/K_p = \frac{n_i}{n_p} \left( \frac{p_{\text{inj}}^i}{p_{\text{inj}}^p} \right)^{\gamma-1} \approx (n_i/n_p) \cdot (m_i/m_p)^{(\gamma-1)/2} \tag{25}$$

and

$$K_p/K_e = (n_p/n_e) \left( \frac{E_{\text{inj}}^2 + 2m_p c^2 E_{\text{inj}}}{E_{\text{inj}}^2 + 2m_e c^2 E_{\text{inj}}} \right)^{(\gamma-1)/2} \approx (n_p/n_e) \cdot \left( \frac{2m_p c^2 E_{\text{inj}}}{E_{\text{inj}}^2 + 2m_e c^2 E_{\text{inj}}} \right)^{(\gamma-1)/2}. \tag{26}$$

Equation (24) has the correct limit (3) when  $E_{\text{inj}} \ll m_e c^2 \ll m_p c^2$ . From Figure 1 it can be seen that for low  $E_{\text{inj}}$  cosmic rays energy density is almost constant (independent of  $E_{\text{inj}}$ ) and usage of equation (3) is justified. When shock velocity can be estimated one should calculate injection energy  $E_{\text{inj}} \approx 4\frac{1}{2}m_p v_s^2$  and use equation (24). Formulae (12) and (13) for magnetic field and minimum energy remain the same.<sup>3</sup> In Figure 2 we give proton to electron energy density ratio as a function of injection energy in our approximation compared to the same data from Bell (1978b). Agreement is quite good despite the approximative character of our formulae.

We have implemented our modified equipartition calculation by developing a PHP code.<sup>4</sup> The code uses some 'typical' starting values for radio spectral index, frequency, flux density, distance, angular radius, filling factor, shock velocity and abundances, which all can be changed or left as such. For example, if shell thickness relative to SNR radius  $\delta$  can be

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<sup>3</sup>Note that  $\kappa$  is no longer ions to electrons energy ratio but a suitable parameter introduced to make new formulae same as the old ones.

<sup>4</sup>The calculator is available at <http://poincare.matf.bg.ac.rs/~arbo/eqp/>.



measured the volume filling factor is  $f = 1 - (1 - \delta)^3$ . Otherwise a typical value  $f = 0.25$  can be used (shell thickness of about 10 %). If shock velocity is unknown one should leave 0 (and simpler equipartition calculation will be performed by using equation (3)). Simple ISM abundances are assumed for start (H:He ratio 10:1). In the implementation of our calculation we used an approximation for the Gamma function (Nemes 2010):

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}} \left( \frac{1}{e} \left( z + \frac{1}{12z - \frac{1}{10z}} \right) \right)^z. \quad (27)$$

### 3. Discussion

From the mathematical point of view, the equipartition calculation is the problem of solving a system of two independent equations (the synchrotron emissivity equation (4) and equation for the total energy in a source (7)) for the three unknown variables (the total energy  $E$ , energy contained in the cosmic rays  $E_{\text{CR}}$  (or  $K_e$ ), and energy contained in the magnetic field  $E_B$  (or  $B$ )). This problem is, of course, impossible to solve without additional assumption. The primary assumption is to seek for the minimum of the total energy of the synchrotron source. Differentiation of equation (8) make that the total energy disappears as unknown variable and two starting equations ((4) and (9)) can now give us solutions for both remaining unknown variables ( $K_e$  and  $B$ ). As the result of differentiation of equation (8), the exact equipartition between energies contained in the magnetic field and cosmic rays is only approximately fulfilled (equation (13)). The alternative assumption, commonly adopted, is the equipartition between energy contained in the cosmic rays and in the magnetic field ( $\epsilon_{\text{CR}} = \epsilon_B$ ). By assuming this we directly link  $K_e$  and  $B$  (see equations (2) and (7)). It is often the case in the literature that these two calculations are commonly referred as either equipartition or the minimum energy calculation. Here, we would like to emphasize that strict equipartition does not have to be assumed for doing the calculation – if  $\epsilon_B/\epsilon_{\text{CR}} = \beta = \text{const}$  is somehow known (independent information about CRs electrons can come from X-ray data (Inverse Compton effect), or about CRs from gamma rays (bremsstrahlung or pion decay)), system can be solved! It means that the magnetic field energy density can be any constant fraction of the cosmic ray energy density and the "equipartition" procedure will give appropriate formulae for the estimation of the amount of the total energy in a source and magnetic field strength, namely

$$B' = \left( \frac{4\beta}{\gamma + 1} \right)^{2/(\gamma+5)} B, \quad (28)$$

where  $B'$  is recalculated field for  $\beta=\text{const}$ , while  $B$  is the field corresponding to the minimum of energy. Total energy calculated in this way is always higher than the minimal energy

obtained from the equipartition i.e. the minimum energy calculation, but magnetic field can be either larger or smaller.

Given the above, the equipartition calculation is not a precise method for the determination of the magnetic field strength, but we can surely estimate its order of magnitude (Duric 1990). The main question is whether there is a physical relation between  $K_e$  and  $B$ ? From Bell’s (1978b) theory,  $K_e$  depends on the CR energy density  $\epsilon_{\text{CR}}$ , injection energy  $E_{\text{inj}}$  and the energy spectral index of cosmic ray particles  $\gamma$ . Thus, implicitly, it must depend on shock velocity which itself depends on time  $t$  or radius  $R = R(t)$  of an SNR. If there is evolution of the magnetic field  $B = B(t)$ ,  $K_e$  and  $B$  must be related. Additionally, in the advanced model of DSA a significant fraction of shock energy is transferred to CRs so the cosmic ray pressure has to be included in equations (Drury 1983). From this, the so-called non-linear DSA theory, the strong magnetic field amplification (approximately two order of magnitudes) is expected especially in the early free expansion phase of SNR evolution, when the very strong shock waves exist (Bell 2004). The non-linear effects thus make efficient cosmic ray acceleration and, at the same time, the significant amplification of the magnetic field strength. This increasing trend for the both energy constituents of the synchrotron emission, again leads to some form of a non-strict equipartition.

The derivation procedure presented in Subsection 2.1, where integration limits for momenta are from 0 to  $\infty$ , leads to equation (12). Using this equation and equation (3), the calculated values of the magnetic field strength are slightly overestimated (a few percent or more, depending on  $E_{\text{inj}}$ ). On the other hand, we neglect all kind of energy losses in this paper. The main processes responsible for the energy losses of the relativistic electrons are the synchrotron radiation and the inverse Compton scattering. These energy losses become significant for electrons especially at the very high energies (radiation power for both processes depends on the square of the electron kinetic energy). The energy losses of electrons result in underestimation of the equipartition magnetic field strength. Thus, in our ”simple approach” (Subsection 2.1), the effects of extending integration limits and energy losses work in the opposite directions and may roughly cancel each other. If integration limits are from  $p_{\text{inj}}$  to  $\infty$  (Subsection 2.2), the equipartition calculation is derived correctly (without assumption about the low shock velocity), but the problem of the energy losses remains and the equipartition estimates fail for the electrons at the highest energies (BK05). This discussion is concentrated only on the energy losses of cosmic ray electrons. The energy pool of cosmic rays is mainly filled with protons and heavier particles which do not lose energy heavily by synchrotron radiation and by inverse Compton scattering. Following Bell’s (1978b) theory, the energy ratio between cosmic ray protons and electrons, for the energy spectral index  $\gamma = 2$ , is approximately 40. If we take  $\gamma = 2.5$ , which is assumed for the obtained curves presented in Figure 2, this ratio is  $\approx 7$ . Due to this, the total cosmic ray

energy losses, in the first approximation, can be neglected, especially for the objects with harder spectra (SNRs), where the energy indices are lower<sup>5</sup>. However, the injection theory has been developed for protons and heavy particles, but not for electrons which may or may not follow the protons. Hence Bell’s formula may give only lower limits for the proton to electron ratio at high energies and hence for the field strength.

In Table 1, we present values of the magnetic field strength and the minimal energy for the sample of 30 Galactic SNRs for which all data<sup>6</sup> necessary for the calculation can be found in the literature. The calculated magnetic field strengths are close to those calculated by using revised equipartition (BK05) and higher than those calculated by using classical equipartition (P70) for all 30 SNRs (see Figures 3 and 4). For P70 calculation we used  $\mathcal{K} = (m_p/m_e)^{(3-\gamma)/2}$ ,  $f = 0.25$  and frequency interval  $10^7 \text{ Hz} < \nu < 10^{11} \text{ Hz}$ . For BK05 calculation, in order to convert from specific intensity to flux density we used  $\frac{I_\nu}{l} = \frac{L_\nu}{4\pi V}$  ( $L_\nu = 4\pi d^2 S_\nu$ ),<sup>7</sup>  $\mathbf{K}_0 = (m_p/m_e)^\alpha$  and  $f = 0.25$ . For five younger Galactic SNRs, for which the forward shock velocities are known, we use general equation for  $\kappa$  (24). Differences between calculated values, obtained by using general and simple approaches, are generally not so high. If we define the fractional error

$$\varphi = \frac{|B - B_{v_s=0}|}{B_{v_s=0}}, \quad (29)$$

for the five SNRs with estimated shock velocities  $\bar{\varphi} = 11\%$ . For the youngest Galactic SNR G1.9+0.3, the fractional error is the largest,  $\varphi_{\text{max}} = 30\%$  (see Figure 4). Further inspection of Table 1, and Figures 3 and 4 leads to the conclusion that variation in abundances of CR species does not significantly alter the final equipartition results.

## 4. Conclusions

In this paper we derived modified equipartition i.e. minimum-energy formula for estimating magnetic fields in supernova remnants. Our approach is similar to BK05 in a sense that we do not integrate over frequencies as P70, however,

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<sup>5</sup>The average energy index for SNRs is  $\gamma \approx 2$  (the radio spectral index  $\alpha \approx 0.5$ ).

<sup>6</sup>Including the distances to SNRs independent of the  $\Sigma - D$  relation (see Urošević et al. 2010 and references therein), and spectral indices  $0.5 < \alpha < 1$ .

<sup>7</sup>At the end of p.415 of their paper, BK05 suggested replacing  $\frac{L}{l}$  with  $\frac{L}{V}$  which is incorrect,  $4\pi$  is missing in denominator of the latter expression.

- (i) we assume power-law spectra  $n(p) \propto p^{-\gamma}$  and integrate over momentum to obtain energy densities of particles,
- (ii) we take into account different ion species and not just equal number of protons and electrons at injection (e.g. for H to He ratio 10:1 there is more energy in  $\alpha$ -particles than in electrons),
- (iii) we use flux density at a given frequency and also assume isotropic distribution of the pitch angles for the remnant as a whole,
- (iv) by incorporating the dependence  $\epsilon = \epsilon(E_{\text{inj}})$  we made the formula applicable to the younger remnants as well.
- (v) we calculate the magnetic field strengths for the sample of 30 Galactic SNRs and obtain values which are close to those calculated by using revised equipartition (BK05) and higher than those calculated by using classical equipartition (P70).

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## REFERENCES

- Arbutina B., Urošević D., Andjelić M., Pavlović M., 2011, MmSAI, in press
- Beck R., 2011, AIPC, 1381, 117
- Beck R., Krause M., 2005, AN, 326, 414 (BK05)
- Bell A. R., 1978a, MNRAS, 182, 147
- Bell A. R., 1978b, MNRAS, 182, 443
- Bell A. R., 2004, MNRAS, 353, 550
- Carlton A.K., Borkowski K., Reynolds S.P., 2011, ApJ, 737, 22
- Drury L. O'C., 1983, Rep. Prog. Phys., 46, 973

- Duric N., 1990, IAUS, 140, 235
- Ghavamian P., Winkler P.F., Raymond J.C., Long K.S., 2002, ApJ, 572, 888
- Govoni F., Feretti L., 2004, IJMPD, 13, 1549
- Green D. A., 2009, BASI, 37, 45
- Hayato A., Yamaguchi H., Tamagawa T. et al., 2010, ApJ, 725, 894
- Heald G., 2009, IAUS, 259, 591
- Longair M. S., 1994, High Energy Astrophysics Vol. 2, Cambridge: Cambridge Univ. Press
- Nemes G., 2010, Archiv der Mathematik 95, 161
- Pacholczyk A. G., 1970, Radio Astrophysics, San Francisco: Freeman and Co. (P70)
- Patnaude D.J., Fesen R.A., 2009, ApJ, 697, 535
- Reynolds S. P., Gaensler B. M., Bocchino F., 2011, SSRv, in press
- Sankrit R., Blair W. P., Delaney T., 2005, AdSpR, 35, 1027
- Urošević D., Vukotić B., Arbutina B., Sarevska M., 2010, ApJ, 719, 950

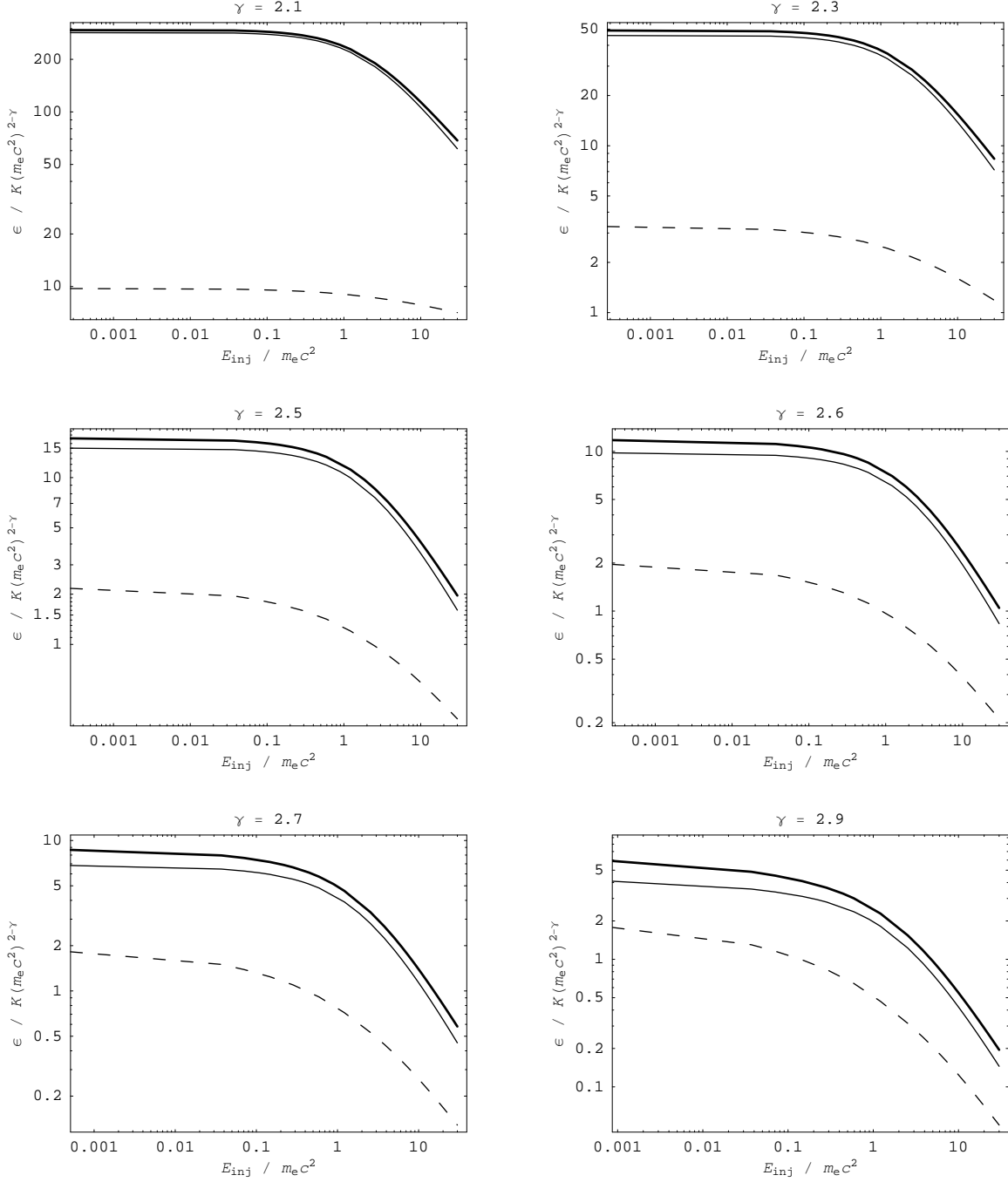


Fig. 1.— CR energy density of ions (H:He = 10:1, solid line), electrons (dashed line) and total (thick solid line) as a function of injection energy, in our approximation.

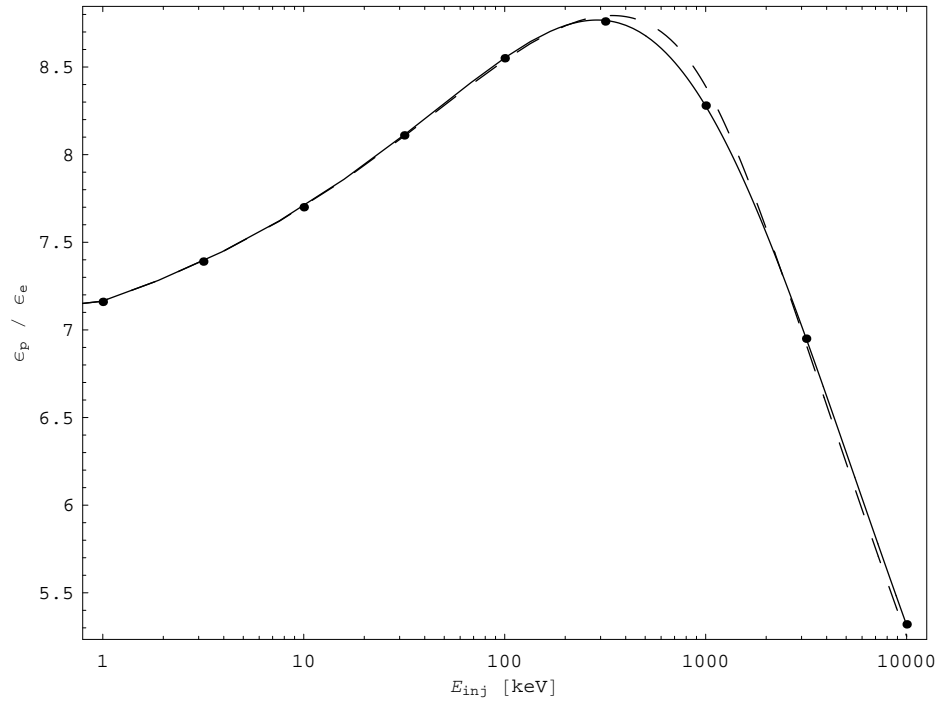


Fig. 2.— Proton to electron energy density ratio as a function of injection energy in our approximation (dashed line) and exact ratio (solid line) for  $\gamma = 2.5$ . Data points are from Bell (1978b).

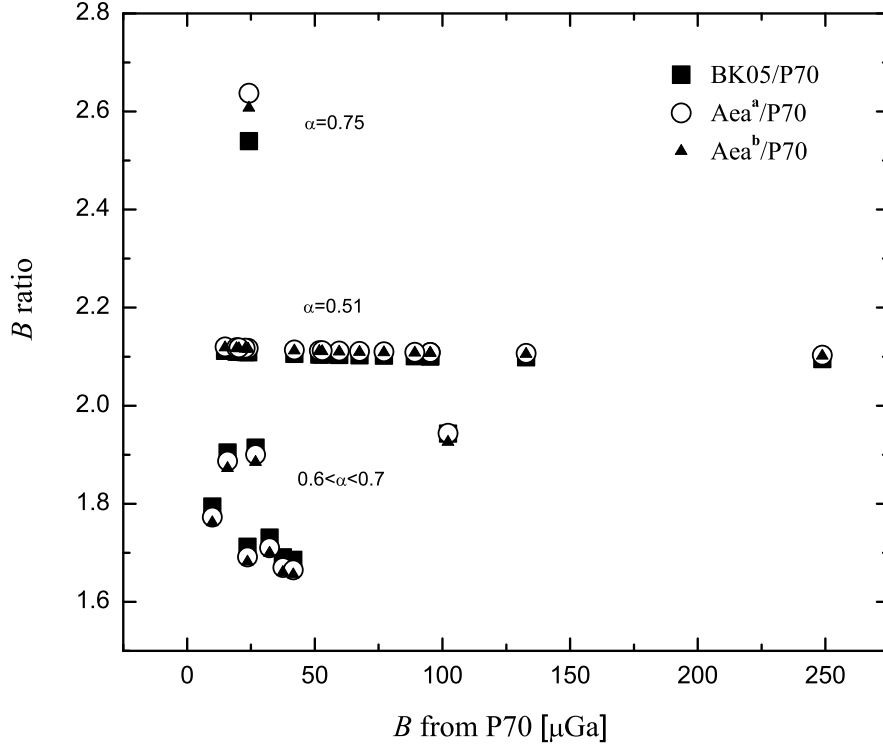


Fig. 3.— Comparison between different calculations for the minimum-energy magnetic field strength ( $B$ ). "  $B$  ratio" represents ratio between BK05 or this paper calculations, and classical equipartition results (P70). Used abbreviations: Aea<sup>a</sup> - this paper (Arbutina et al.), simple approach for  $p^+:e^-=1:1$ ; Aea<sup>b</sup> - this paper, simple approach for  $H:He=10:1$ . Data are from Table 1 (25 SNRs).



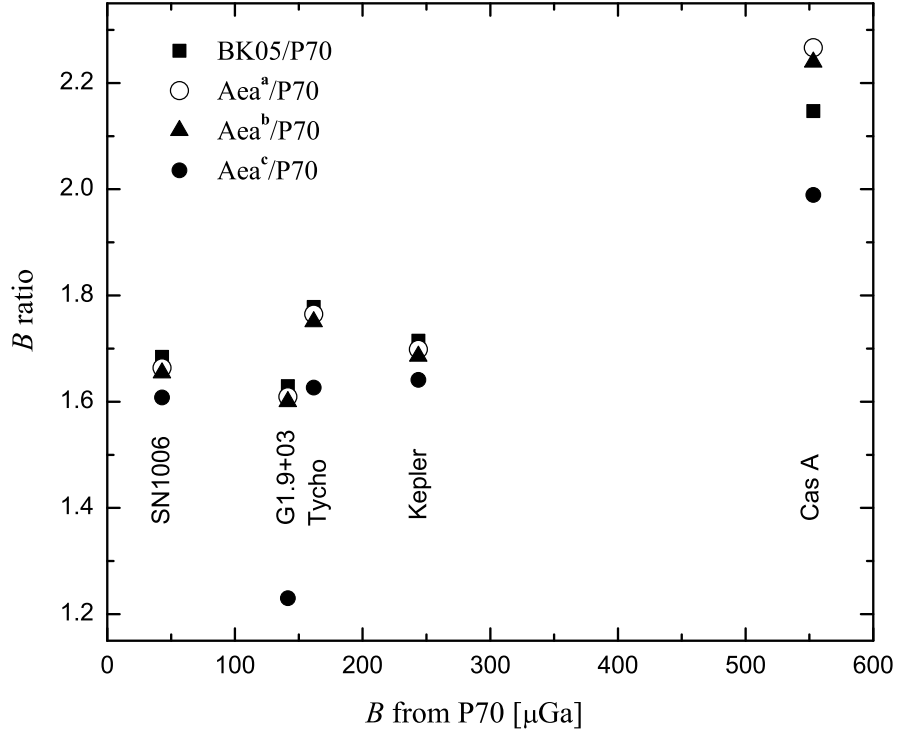


Fig. 4.— Comparison between different calculations for the minimum-energy magnetic field strength ( $B$ ) for 5 young SNRs with available forward shock velocities. "B ratio" represents ratio between BK05 or this paper calculations, and classical equipartition results (P70). Used abbreviations: Aea<sup>a</sup> - this paper (Arbutina et al.), simple approach for  $p^+:e^-=1:1$ ; Aea<sup>b</sup> - this paper, simple approach for  $H:He=10:1$ ; Aea<sup>c</sup> - this paper, general approach for  $H:He=10:1$ . Data are from Table 1 (5 SNRs).

Table 1. Calculated magnetic field strengths and total energies for sample of 30 Galactic SNRs

Name <sup>d</sup>	Other names	Pacholczyk(1970)		Beck & Krause (2005)		This paper <sup>a</sup>		This paper <sup>b</sup>		This paper <sup>c</sup>	
		$B$	$E_{\min}$	$B$	$E_{\min}$	$B$	$E_{\min}$	$B$	$E_{\min}$	$B$	$E_{\min}$
G4.5+6.8 <sup>e</sup>	Kepler, SN1604, 3C358	2.44E-04	8.38E+47	4.18E-04	2.34E+48	4.14E-04	2.30E+48	4.11E-04	2.26E+48	4.00E-04	2.14E+48
G21.80.6	Kes 69	7.71E-05	1.10E+50	1.62E-04	4.82E+50	1.63E-04	4.86E+50	1.63E-04	4.86E+50	-	-
G23.30.3	W41	6.75E-05	5.86E+49	1.42E-04	2.57E+50	1.43E-04	2.59E+50	1.42E-04	2.59E+50	-	-
G27.4+0.0	4C04.71	1.02E-04	3.75E+48	1.99E-04	1.32E+49	1.99E-04	1.33E+49	1.97E-04	1.30E+49	-	-
G33.6+0.1	Kes 79, 4C00.70, HC13	9.52E-05	3.79E+49	2.00E-04	1.66E+50	2.01E-04	1.68E+50	2.01E-04	1.67E+50	-	-
G46.80.3	HC30	5.96E-05	4.88E+49	1.25E-04	2.14E+50	1.26E-04	2.16E+50	1.26E-04	2.16E+50	-	-
G53.62.2	3C400.2, NRAO 611	2.42E-05	3.18E+48	6.14E-05	1.88E+49	6.38E-05	2.02E+49	6.30E-05	1.98E+49	-	-
G65.1+0.6	-	9.90E-06	1.90E+50	1.78E-05	5.87E+50	1.76E-05	5.73E+50	1.74E-05	5.66E+50	-	-
G93.70.2	CTB 104A, DA 551	2.68E-05	1.09E+49	5.13E-05	3.80E+49	5.09E-05	3.74E+49	5.05E-05	3.68E+49	-	-
G96.0+2.0	-	1.49E-05	2.20E+48	3.15E-05	9.74E+48	3.16E-05	9.82E+48	3.16E-05	9.81E+48	-	-
G108.20.6	-	1.94E-05	2.52E+49	4.09E-05	1.12E+50	4.11E-05	1.13E+50	4.11E-05	1.12E+50	-	-
G109.11.0	CTB 109	5.18E-05	1.40E+49	1.09E-04	6.16E+49	1.09E-04	6.21E+49	1.09E-04	6.20E+49	-	-
G111.72.1 <sup>f</sup>	Cassiopeia A, 3C461	5.53E-04	1.32E+49	1.19E-03	5.56E+49	1.25E-03	6.19E+49	1.24E-03	6.05E+49	1.10E-03	4.76E+49
G114.3+0.3	-	2.40E-05	6.05E+47	5.05E-05	2.67E+48	5.07E-05	2.69E+48	5.07E-05	2.69E+48	-	-
G116.5+1.1	-	2.27E-05	6.21E+48	4.80E-05	2.75E+49	4.82E-05	2.77E+49	4.81E-05	2.76E+49	-	-
G116.9+0.2	CTB 1	3.23E-05	1.48E+48	5.60E-05	4.26E+48	5.53E-05	4.16E+48	5.49E-05	4.10E+48	-	-
G120.1+1.4 <sup>g</sup>	Tycho, 3C10, SN1572	1.62E-04	1.63E+48	2.88E-04	4.88E+48	2.85E-04	4.80E+48	2.83E-04	4.73E+48	2.63E-04	4.09E+48
G132.7+1.3	HB3	2.36E-05	2.69E+49	4.05E-05	7.58E+49	4.00E-05	7.39E+49	3.98E-05	7.31E+49	-	-
G160.9+2.6	HB9	1.58E-05	3.08E+50	3.02E-05	1.06E+51	2.99E-05	1.04E+51	2.97E-05	1.03E+51	-	-
G205.5+0.5	Monoceros Nebula	2.03E-05	6.65E+49	4.27E-05	2.94E+50	4.29E-05	2.97E+50	4.29E-05	2.96E+50	-	-
G260.43.4	Puppis A, MSH 0844	5.29E-05	4.31E+49	1.11E-04	1.90E+50	1.12E-04	1.91E+50	1.12E-04	1.91E+50	-	-
G292.20.5	-	4.20E-05	4.78E+49	8.84E-05	2.11E+50	8.87E-05	2.12E+50	8.87E-05	2.12E+50	-	-
G296.80.3	115662	3.75E-05	6.50E+49	6.34E-05	1.41E+50	6.26E-05	1.38E+50	6.22E-05	1.36E+50	-	-
G304.6+0.1	Kes 17	9.52E-05	3.73E+49	2.00E-04	1.64E+50	2.01E-04	1.65E+50	2.01E-04	1.65E+50	-	-
G315.42.3	RCW 86, MSH 1463	4.16E-05	1.37E+49	7.01E-05	3.75E+49	6.92E-05	3.66E+49	6.88E-05	3.62E+49	-	-
G327.6+14.6 <sup>h</sup>	SN1006, PKS 145941	4.28E-05	4.65E+48	7.22E-05	1.27E+49	7.13E-05	1.24E+49	7.09E-05	1.22E+49	6.89E-05	1.16E+49
G332.40.4	RCW 103	1.33E-04	4.63E+48	2.79E-04	2.03E+49	2.80E-04	2.04E+49	2.80E-04	2.04E+49	-	-
G337.80.1	Kes 41	8.92E-05	6.80E+49	1.87E-04	2.99E+50	1.88E-04	3.01E+50	1.88E-04	3.00E+50	-	-
G349.7+0.2	-	2.49E-04	6.49E+49	5.21E-04	2.83E+50	5.23E-04	2.85E+50	5.23E-04	2.85E+50	-	-
G1.9+03 <sup>i</sup>	-	1.41E-04	3.65E+47	2.30E-04	9.33E+47	2.28E-04	9.11E+47	2.26E-04	9.00E+47	1.74E-04	5.31E+47

Note. — All units are in CGS system.  $B$  is magnetic field strength calculated for minimum-energy assumption.

<sup>a</sup>Simple approach for  $p^+:e^-=1:1$ .

<sup>b</sup>Simple approach for  $H:He=10:1$ .

<sup>c</sup>General approach for  $H:He=10:1$ , for young SNRs with available forward shock velocities ( $v_s$ ).

<sup>d</sup>According to Green's (2009) catalogue from which data for SNRs, except shock velocities, has been taken.

<sup>e</sup> $v_s = 1660\text{km/s}$  (Sankrit et al. 2005).

<sup>f</sup> $v_s = 4900\text{km/s}$  (Patnaude et al. 2009).

<sup>g</sup> $v_s = 4700\text{km/s}$  (Hayato et al. 2011).

<sup>h</sup> $v_s = 2890\text{km/s}$  (Ghavamian et al. 2002).

<sup>i</sup> $v_s = 14000\text{km/s}$  (Carlton et al. 2011).